Using L^p spaces to measure uncertainty in insurance premiums

Enrique Calderín Ojeda[a], **Emilio Gómez Déniz**[b] **and Ignacio J. Cabrera Ortega**[c]

[a] *Assistant Researcher from Department of Quantitative Methods in Economics, University of Las Palmas de Gran Canaria, 35017-Las Palmas, Spain.*

[b] *Department of Mathematics, University of Las Palmas de Gran Canaria, 35017-Las Palmas de Gran Canaria, Spain*

[c] *Department of Mathematics, University of Las Palmas de Gran Canaria, 35017-Las Palmas de Gran Canaria, Spain*

In this paper we consider statistical problems arising from applications concerning insurance-premium calculation. We describe an integrated set of Bayesian tools for modelling premiums systems using the local approach. In this article the local approach is broadly defined, ranging from differentiation of functions to differentiation of functionals, reviewing some of the basic formulations for local assessment of prior influence. We then discuss the use of the local analysis to study sensitivity concerning insurance-premium calculation.
Key words: Premium, Local robustness, Fréchet Derivative, Norm, Poisson–Gamma model.

1 Introduction

Questions concerning robustness are very common in statistical analysis. We need to know how strongly the answer coming out depends on the elements going in. For a bayesian analysis these elements are the data, the model, the prior and the loss function. The output might be the posterior distribution. The study of robustness try to answer how sensitive is the output to the input. In this paper we focus on prior influence; we will study the rate of change of the posterior distribution with respect to infinitesimal changes in the prior distribution. The local sensitivity analysis consists of the use of differential calculus to asses sensitivity. Furthermore, local sensitivity can be useful when we want to compare the sensitivity of several posterior quantities of interest.

The most commonly premium principles consists of assuming that the individual risk for the number of claims has a given distribution depending on a parameter $\theta \in \Theta$ which is distributed according to a prior distribution $\pi_0(\theta)$. Assuming that the risks are independent we can compute the premium using the expression

$$\mathcal{P}(\pi_0) = \frac{E_{\pi_0^{\mathbf{k}}}[g(\theta)]}{E_{\pi_0^{\mathbf{k}}}[h(\theta)]}, \tag{1}$$

where $\pi_0^{\mathbf{k}}$ is the posterior distribution of π_0 after observing the data $\mathbf{k}$ and $g(\theta)$, and $h(\theta)$ are appropriate functions whose expectations under π_0 exist.

Heilmann (1988) derives premium principles using decision theory under loss function in the form $L(k, \mathcal{P}) = g(k)[h(k) - h(\mathcal{P})]^2$. From different types of functions for $g(k)$ and $h(k)$, many premium principles using decision theory and hence diverse risk premiums can be obtained. For example,for $g(k) = k$ and $h(k) = 1$ we obtain the net premium principle (Eichenauer et al. 1988; Gómez et al.2002; Heilmann, 1989 and Lemaire, 1995; among others) and for $g(k) = k^2$ and $h(k) = k$ the variance principle (Heilmann, 1989; Lemaire, 1995 and Gomez et al.2002 among others).

The risk premium, $\mathcal{P}(\theta)$, is usually obtained by minimizing $E_{f(k|\theta)}[L(k, \mathcal{P}(\theta))]$ applying the same loss function as that used to obtain the Bayes premium, where $f(k \mid \theta)$ is the probability density function of claims.

The paper is organized as follows. In Section 2, we define some previous results, here the Fréchet Derivative of functionals and some features of a size functional are introduced. In Section 3, we comment on the study of local robustness, we will study the Fréchet derivative of the quotient of posterior expectations by using an expression of the difference of the quotients of posterior expectation. In section 4, we discuss the interpretation of the norm of the derivatives. In section 5, we compute some numerical illustrations based on a Poisson–Gamma model. Finally, in section 6 we will comment on conclusions and future studies with relation to this subject.

2 Previous results

Robustness of the prior distribution is considered in this section. The rate of change in the inference with respect to change in the prior; this topic is known as local or infinitesimal robustness.

Consider a measure space $(\Theta, \mathcal{B}, \mu)$. Let π_0 be a probability density with respect to $(\Theta, \mathcal{B}, \mu)$, which induces a probability measure Π_0. A simple way of incorporating the Bayesian local study in a parametric model for the sample space K and the parameter space $\Theta \subset \mathbb{R}$, $\{f(k \mid \theta), \theta \in \Theta\}$, is through a prior distribution $\pi(\theta)$ for the parameter θ such that $\Gamma = \{\pi : \pi = \pi_0 + u\}$, where u is a signed measure with $u(\Theta) = 0$ with Θ_π. When the signed measure u is in the form $u = \varepsilon(\widehat{q} - \pi_0)$ with $\widehat{q} \in \mathcal{Q}$, $\varepsilon \in [0, 1]$ we have the well known ε–contaminated class (Martín *et al.* (2003), Gustafson (1996) and Gómez *et al.* (2002); among others). Observe that the perturbation on π_0 is linear.

To assess sensitivity we can quantify the worst case of sensitivity by computing the norm of (1) when π_0 is replaced by $\pi \in \Gamma$. This can be interpreted as the maximum change of the quotient of posterior expectations to prior discrepancy. The question now is how to quantify the magnitude of the discrepancy or perturbation. Gustafson (1996) proposes to use the size of u which is given by $\text{size}\,(u) = \|u/\pi_0; \Pi_0\|_p = \left(\int_\Theta (u/\pi_0)^p \, d\Pi_0\right)^{1/p}$, if $p < \infty$ and $\text{size}\,(u) = \|u/\pi_0; \Pi_0\|_p = \text{ess}\sup_\Theta u/\pi_0$, if $p = \infty$, being $p \in [0,1]$, $q \in [0,1]$ the solution to $p + q = pq$. The size is chosen in this form because (see Gustafson (1996)) satisfies the following desirable axioms: 1) The size is a norm; 2) The size is invariant under a change of dominating measure; 3) The size is invariant under transformation; and 4) The size is finite, which guarantees that π is integrable.

Now we can choose Γ for $c > 0$ in the following form

$$\Gamma = \{\pi : \|u/\pi_0; \Pi_0\|_p \leq c\},$$

the class of all limited perturbations of π_0.

Local robustness uses differential calculus to assess sensitivity. Then if $\mathcal{P}(\pi_0)$ is perturbed by u, then it is natural to use $\dot{\mathcal{P}}(u)$ to quantify prior influence. To do this we will compute the norm of the Fréchet derivative of $\dot{\mathcal{P}}(u)$ with respect to the prior (Milne (1980), Diaconis and Freedman (1986), Gustafson (1996) and Martín *et al.* (2003), among others).

Consider an operator $\mathcal{P} : \mathcal{U} \to \mathcal{V}$ where $\mathcal{U}$ and $\mathcal{V}$ are normed vector spaces, $\mathcal{L}(\mathcal{U}, \mathcal{V})$ the space of bounded linear transformations $\mathcal{U} \to \mathcal{V}$ and some open subset $\mathcal{D}(T)$ of $\mathcal{U}$. The following definition of the Fréchet derivative appears in Milne (1980).

Definition 1. *An operator $\mathcal{P} : \mathcal{U} \to \mathcal{V}$ is Fréchet differentiable at $x \in \mathcal{D}(T) \subset \mathcal{U}$ if there exists a continuous linear operator $\dot{\mathcal{P}}(x) \in \mathcal{L}(\mathcal{U}, \mathcal{V})$ satisfying*

$$\|\mathcal{P}(u_1 + u_2) - \mathcal{P}(u_1) - \dot{\mathcal{P}}(u_1)u_2\|_{\mathcal{V}} = o(\|u_2\|_{\mathcal{U}})$$

The operator $\dot{\mathcal{P}}(u_1)$ is called the Fréchet or strong derivative of $\mathcal{P}$ at u_1.

The norm of a Fréchet derivative is given by

$$\left\|\dot{\mathcal{P}}(u_1)\right\| = \sup_{u_2 \neq 0} \frac{\left\|\dot{\mathcal{P}}(u_1)u_2\right\|_{\mathcal{V}}}{\|u_2\|_{\mathcal{U}}} = \sup_{\|u_2\| \leq 1} \frac{\left\|\dot{\mathcal{P}}(u_1)u_2\right\|_{\mathcal{V}}}{\|u_2\|_{\mathcal{U}}} = \sup_{\|u_2\| = 1} \left\|\dot{\mathcal{P}}(u_1)u_2\right\|_{\mathcal{V}}$$

To end this section we can remember, because it will be useful later, the well known result in functional analysis that for any map I from Ω to $\mathbb{R}$ we can write $I = I^+ - I^-$ and $|I| = I^+ + I^-$, where $I^+ = \max(I, 0)$ and $I^- = \max(-I, 0)$

3 Local robustness study

The first element for an analysis of sensitivity to the prior is a framework for considering deviations from the base prior. We will consider a framework that uses additive or linear contaminations of the prior.

We will consider from now on, a parameter space $\Theta \subset \mathbb{R}$ for which (Θ, β, μ) is a measure space where μ is Lebesgue measure. Let π_0 be a probability density with respect to $\Theta \subset \mathbb{R}$ inducing a probability measure Π_0; π_0 is referred to as the base prior.
A straightforward way to construct densities close to π_0 is by perturbation, and the most common form of perturbation is linear. The result of perturbation is denoted by:

$$\pi(\theta) = \pi_0(\theta) + u(\theta)$$

Consider observance of data x giving rise to likelihood function $f(x \mid \theta)$. The functions of the parameter that are of inferential interest are denoted as $g(\theta)$ and $h(\theta)$.
Let $\Gamma = \{\pi : \|u/\pi_0; \Pi_0\|_p \leq c\}$, and $\mathcal{V} = \mathbb{R}$. We will adopt $\|u/\pi_0; \Pi_0\|_p$ as the norm on $\mathcal{U}$ and the absolute value as the norm on $\mathbb{R}$.

Let the mapping $\mathcal{P}(u) : \Theta \longrightarrow \mathbb{R}$ be the quotient of posterior expectations of measurable functions g and h :

$$\mathcal{P}(u) = \frac{\displaystyle\int_\Theta g(\theta)\pi(\theta)f(x \mid \theta)d\mu(\theta)}{\displaystyle\int_\Theta h(\theta)\pi(\theta)f(x \mid \theta)d\mu(\theta)} = \frac{N_\pi^g}{N_\pi^h} \tag{2}$$

Let $\dot{\mathcal{P}}(u_0)u$ the operator Fréchet derivative of $\mathcal{P}(u)$ at u_0 in the direction u.The next theorem compute the derivative of (2)

Theorem 1. *Suppose that gf and hf are bounded. Then*

$$\dot{\mathcal{P}}(u_0)u = \frac{1}{N_{\pi_0}^h}\left(N_u^g - \frac{N_{\pi_0}^g}{N_{\pi_0}^h}N_u^h\right) \tag{3}$$

Proof.-

$$\mathcal{P}(\pi_0 + u) - \mathcal{P}(\pi_0) = \frac{N_{\pi_0+u}^g}{N_{\pi_0+u}^h} - \frac{N^g\pi_0}{N^h\pi_0} = \frac{N_{\pi_0}^g + N_u^g}{N^h\pi_0 + N_u^h} - \frac{N_{\pi_0}^g}{N_{\pi_0}^h}$$

$$= \frac{1}{N_{\pi_0}^h}\left(\frac{N_{\pi_0}^g N_{\pi_0}^h + N_u^g N^h\pi_0}{N_{\pi_0}^h + N_u^h} - N_{\pi_0}^g\right)$$

$$
\begin{aligned}
&= \frac{1}{N^h_{\pi_0}} \left(\frac{N^g_{\pi_0} N^h_{\pi_0} + N^g_u N^h_{\pi_0} - N^h_{\pi_0} N^g_{\pi_0} - N^g_{\pi_0} N^h_u}{N^h_{\pi_0} + N^h_u} \right) \\
&= \frac{1}{N^h_{\pi_0}} \left(\frac{N^g_u N^h_{\pi_0} - N^g_{\pi_0} N^h_u \frac{N^h_{\pi_0}}{N^h_{\pi_0}} + N^g_u N^h_u - N^g_u N^h_u + \frac{N^g_{\pi_0}}{N^h_{\pi_0}} N^h_u N^h_u - \frac{N^g_{\pi_0}}{N^h_{\pi_0}} N^h_u N^h_u}{N^h_{\pi_0} + N^h_u} \right) \\
&= \frac{1}{N^h_{\pi_0}} \left(\frac{N^g_u (N^h_{\pi_0} + N^h_u) - \frac{N^g_{\pi_0}}{N^h_{\pi_0}} (N^h_u N^h_{\pi_0} + N^h_u N^h_u) - N^g_u N^h_u + \frac{N^g_{\pi_0}}{N^h_{\pi_0}} N^h_u N^h_u}{N^h_{\pi_0} + N^h_u} \right) \\
&= \frac{1}{N^h_{\pi_0}} \left(\frac{N^g_u (N^h_{\pi_0} + N^h_u) - \frac{N^g_{\pi_0}}{N^h_{\pi_0}} N^h_u (N^h_{\pi_0} + N^h_u) + \frac{N_{\pi_0}}{N^h_{\pi_0}} N^h_u N^h_u - N^g_u N^h_u}{N^h_{\pi_0} + N^h_u} \right) \\
&= \frac{1}{N^h_{\pi_0}} \left(N^g_u - \frac{N^g_{\pi_0}}{N^h_{\pi_0}} N^h_u \right) - \frac{1}{N^h_{\pi_0}} \frac{N^g_u N^h_u - \frac{N^g_{\pi_0}}{N^h_{\pi_0}} N^h_u N^h_u}{N^h_{\pi_0} + N^h_u} \\
&= \frac{1}{N^h_{\pi_0}} \left(N^g_u - \frac{N^g_{\pi_0}}{N^h_{\pi_0}} N^h_u \right) - \frac{N^h_{\pi_0} N^g_u N^h_u - \frac{N^g_{\pi_0}}{N^h \pi_0} N^h_u N^h_u N^h_{\pi_0}}{(N^h_u)^2 (N^h_{\pi_0} + N^h_u)} \\
&= \frac{1}{N^h_{\pi_0}} \left(N^g_u - \frac{N^g_{\pi_0}}{N^h_{\pi_0}} N^h_u \right) - \frac{N^h_u (N^g_u N^h_{\pi_0} - N^h_u N^g_{\pi_0})}{\left(N^h_{\pi_0}\right)^2 (N^h_{\pi_0} + N^h_u)}
\end{aligned}
$$

□

The second term of the last expression can be rewritten as

$$
R = \frac{N^h_u (N^g_u N^h_{\pi_0} - N^h_u N^g_{\pi_0})}{\left(N^h_{\pi_0}\right)^2 (N^h_{\pi_0} + N^h_u)}
$$

It only remains to show that $\|R\| \to 0$

So we can obtain

$$
\mathcal{P}(\pi_0 + u) - \mathcal{P}(\pi_0) \approx \frac{1}{N^h_{\pi_0}} \left(N^g_u - \frac{N^g_{\pi_0}}{N^h_{\pi_0}} N^h_u \right)
$$

$$
\begin{aligned}
\|R\| &\leq \frac{|N^h_u|}{|N^h_{\pi_0} + N^h_u||N^h_{\pi_0}|} \frac{|N^g_u N^h_{\pi_0} - N^g_{\pi_0} N^h_u|}{N^h_{\pi_0}} \\
&= \frac{|N^h_u|}{|N^h_{\pi_0} + N^h_u||N^h_{\pi_0}|} |N^g_u - \frac{N^g_{\pi_0}}{N^h_{\pi_0}} N^h_u|
\end{aligned}
$$

We may bound the last inequality since hf and gf are bounded

$$|N_u^h| \leq \int_\Theta |h(\theta)||u(\theta|x)|d\mu(\theta) \leq \int_\Theta |h(\theta)||f(x \mid \theta)||u(\theta)|d\mu(\theta)$$
$$\leq \int_\Theta a|u(\theta)|d\mu(\theta) = a\int_\Theta |u(\theta)|d\mu(\theta) = a\|u\|$$

$$|N_u^g| \leq \int_\Theta |g(\theta)||u(\theta|x)|d\mu(\theta) \leq \int_\Theta |g(\theta)||f(x \mid \theta)||u(\theta)|d\mu(\theta)$$
$$\leq \int_\Theta b|u(\theta)|d\mu(\theta) = b\int_\Theta |u(\theta)|d\mu(\theta) = b\|u\|$$

$$\|R\| \leq \frac{a\|u\|}{|N_{\pi_0}^h + N_u^h||N_{\pi_0}^h|}\left|b\|u\| - \frac{N_{\pi_0}^g}{N^h\pi_0}a\|u\|\right| = \frac{a\|u\|^2}{|N_{\pi_0}^h + N_u^h||N_{\pi_0}^h|}\left|b - \frac{N_{\pi_0}^g}{N^h\pi_0}a\right| \leq C_{\pi_0}\|u\|^2 = o(\|u\|)$$

Since C_{π_0} is a constant depending on π_0

$$\|R\| \leq o(\pi_0)$$

if $u \to 0$ then $\|R\| \to 0$.

□

Using different types of functions for $g(\theta)$ and $h(\theta)$ we can obtain different expressions for the quotient of posterior expectations. For example, for $g(\theta) = \theta$ and $h(\theta) = 1$ we get the net premium principle and for $g(\theta) = \theta^2$ and $h(\theta) = \theta$ the variance principle.

We will prove now, that we may interchange in expression (3) the direction u by the direction $\frac{u}{\pi_0}$

Proposition 1. $\dot{\mathcal{P}}(u_0)u = \dot{\mathcal{P}}(u_0)\frac{u}{\pi_0}$ *where* $N_{\frac{u}{\pi_0}}^{(\cdot)} = \int_\Theta (\cdot) f(x \mid \theta)\frac{u}{\pi_0}d\Pi_0$

Proof.- The result follows immediately of Theorem 1 by interchanging in expression (3) u and $d\mu(\theta)$ by $\frac{u}{\pi_0}$ and $d\Pi_0(\theta)$

$$\dot{\mathcal{P}}(u_0)u = \frac{N_u^g}{N_\pi^h} - \frac{N_\pi^g}{N_\pi^h}\frac{N_u^h}{N_\pi^h} = \frac{N_{\frac{u}{\pi_0}}^g}{N_\pi^h} - \frac{N_\pi^g}{N_\pi^h}\frac{N_{\frac{u}{\pi_0}}^h}{N_\pi^h} = \dot{\mathcal{P}}(u_0)\frac{u}{\pi_0}$$

□

4 Computing the norm of the derivative

The norm of the derivative at 0 is of special interest. The norm is the maximum rate of change of the quotient of posterior expectations relative to the prior, as the prior is locally perturbed away from the single prior π_0. The following corollary allows us to compute the norm.

Corollary 1.

$$\left\| \dot{\mathcal{P}}(0) \right\|_p = \sup_{\|u;\mu\|_p = 1} \left| \dot{\mathcal{P}}(0)u \right| = \sup_{\left\| \frac{u}{\pi_0};\Pi_0 \right\|_p = 1} \left| \dot{\mathcal{P}}(0)\frac{u}{\pi_0} \right| \tag{4}$$

Proof.- Clearly holds using Proposition 1.

Lemma 1. *Let* $E^x_{\pi_0}[g(\theta)] = \rho_g$ *and* $E^x_{\pi_0}[h(\theta)] = \rho_h$ *then* $\dot{\mathcal{P}}(u_0)u = \frac{1}{\rho_h}\left[\int_\Theta I(\theta)u(\theta)d\mu(\theta)\right]$ *where*

$$I(\theta) = \left(g(\theta) - h(\theta)\frac{\rho_g}{\rho_h}\right)\frac{\pi^x_0(\theta)}{\pi_0(\theta)}$$

Proof.- Let the mapping $\mathcal{P}(u) : \Theta \longrightarrow \mathbb{R}$ be the quotient of posterior expectations of measurable functions g and h :

$$\mathcal{P}(u) = \frac{\displaystyle\int_\Theta g(\theta)\pi_0(\theta)f(x \mid \theta)d\mu(\theta)}{\displaystyle\int_\Theta h(\theta)\pi_0(\theta)f(x \mid \theta)d\mu(\theta)} = \frac{N^g_{\pi_0}}{N^h_{\pi_0}} \tag{5}$$

We can write the derivative of (5) as follows

$$\dot{\mathcal{P}}(u_0)u = \frac{1}{N^h_{\pi_0}}\left(N^g_u - \frac{N^g_{\pi_0}}{N^h_{\pi_0}}N^h_u\right) = \frac{1}{N^h_{\pi_0}}\left[N^g_u - \frac{N^g_{\pi_0}}{N^h_{\pi_0}}\ N^h_u\right]$$

$$= \frac{\frac{1}{D_{\pi_0}}}{\frac{N^h_{\pi_0}}{D_{\pi_0}}}\left[N^g_u - \frac{N^g_{\pi_0}}{N^h_{\pi_0}}N^h_u\right] = \frac{\frac{1}{D_{\pi_0}}}{\rho_h}\left[N^g_u - \frac{N^g_{\pi_0}}{N^h_{\pi_0}}N^h_u\right]$$

$$= \frac{\frac{1}{D_{\pi_0}}}{\rho_h}\left[N^g_u - \frac{\rho_g}{\rho_h}N^h_u\right] = \frac{\frac{1}{D_{\pi_0}}}{\rho_h}\left[\int_\Theta \left(g(\theta) - h(\theta)\frac{\rho_g}{\rho_h}\right)u(\theta)f(x \mid \theta)d\mu(\theta)\right]$$

$$= \frac{1}{\rho_h}\left[\int_\Theta \left(g(\theta) - h(\theta)\frac{\rho_g}{\rho_h}\right)u(\theta)\frac{f(x \mid \theta)}{D_{\pi_0}}d\mu(\theta)\right]$$

$$= \frac{1}{\rho_h}\left[\int_\Theta \left(g(\theta) - h(\theta)\frac{\rho_g}{\rho_h}\right)\frac{\pi^x_0(\theta)}{\pi_0(\theta)}u(\theta)d\mu(\theta)\right]$$

□

The expression $\frac{1}{\rho_h}I(\theta)$ is called influence function.

Finally, we have the main result of this paper, which give us the norm of the quotient of posterior expectations.

Theorem 2.

$$\left\|\dot{\mathcal{P}}(0)\right\|_p = \max\left\{\left\|\frac{1}{\rho_h}I^+;\Pi_0\right\|_q, \left\|\frac{1}{\rho_h}I^-;\Pi_0\right\|_q\right\},$$

which p,q such that $p+q=pq$.

Proof.- Since this theorem may be considered a generalization of the result 10 in Gustafson (1996a), the proof is omitted.

Corollary 2. *If q is odd and $h(\theta)=1$, then*

$$\left\|\dot{T}^{f,g}(0)\right\|_p = \left\|\frac{1}{\rho_h}I(\theta)\right\|_q$$

Proof.- It is obvious using Theorem 2. □

5 Numerical Ilustration

We assume that the individual risk for the number of claims has a Poisson-type distribution and its mean is distributed as a prior distribution. We will consider a group in which the claim proneness of a risk is represented by a risk parameter θ. We assume that the risks are independent, so we take a risk θ and a assume that the number of claims for each policy holder fits a Poisson distribution with mean $\theta > 0$, $f(k \mid \theta) = e^{-\theta}\theta^k/k!, k = 0, 1, ...,$ whose parameter θ varies from one individual to another, reflecting the individual´s claim propensity, $\theta = E(K|\theta)$. This parameter is assumed to be a random variable and to follow a structure function $\pi_0(\theta)$.

Consider a policyholder, drawn randomly from the insurance portfolio, who is observed to have the sequence of claims $k_1, k_2, ..., k_t$ over t periods. We assume these to be independent and equally distributed. Assuming $\underline{k} = (k_1, k_2, ..., k_t)$ the Bayes premium is defined (Heilmann, 1989) as the real number $P(\underline{k})$ minimizing the posterior expected loss $E_{\pi_0(\theta|k)}[L(P(\theta), P(\underline{k})]$,i.e., the posterior expected loss sustained by a practitioner who takes action $P(\underline{k})$ in stead of $P(\theta)$, the risk premium, which is unknown.

In this paper, we will consider as prior distribution a Gamma distribution. In this case, by combining the likelihood function with the prior distribution, $\mathcal{G}(a,b)$, $\pi_0(\theta) = b^a \theta^{a-1} e^{-b\theta}/\Gamma(a),\ a > 0, b > 0$, the posterior distribution remains a Gamma distribution with the updated parameters $\mathcal{G}(a+k, b+t)$.

In order to show the local robustness of this model we have chosen car insurance data which appear in Bühlmann (1970,pp.107). The data refers to the number of claims made in one year.

We have used the maximum likelihood method to estimate the parameter of the prior distribution. We have obtained these results $\hat{a} = 0.766595$ and $\hat{b} = 3.40513$, and taking into account the well known result that the predictive distribution is a Negative Binomial distribution with parameters a and $b/(b+1)$.

We have plotted the influence functions in Figures 1, 2 and 3 for values of $t = 1, 5, 10$ respectively. Every Figure displays six plots, one for each value of $k = 1, 2, 3, 4, 5, 6$ from left to right and from up to down.

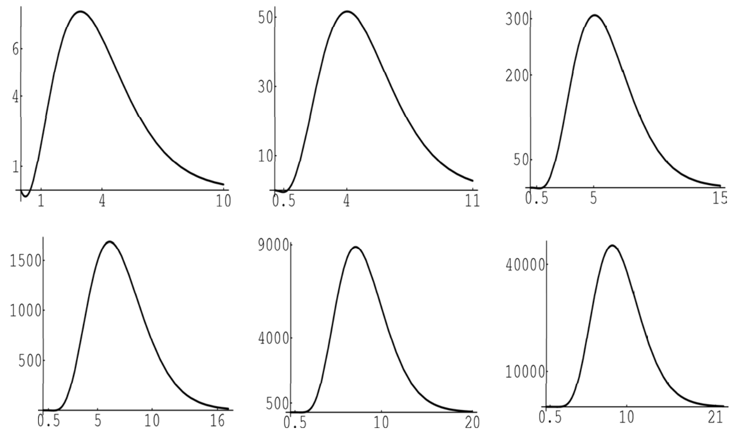

Figure 1: Influence functions for t=1

All the functions take on a similar shape as a function of the periods and the number of claims, but exhibit a stronger dependence on the parameter θ. The influence function is asymptotically diminishing with larger values of the parameter.

In Figures 4, 5 and 6, we have plotted the norms of the derivatives of the premiums. In our case , the norms of the derivatives for $p = 2, \infty$ have been computed by one-dimensional numerical integration and for $p = 1$ we have used maximization to calculate the norms of the derivatives.

In all cases, that is, for each value of p, we observe lack of robustness when the average number of claims increases. We have obtained the most robust results when $k = 0$. We can also observe that the increase of the norm is weaker when $p = \infty$ and $q = 1$.

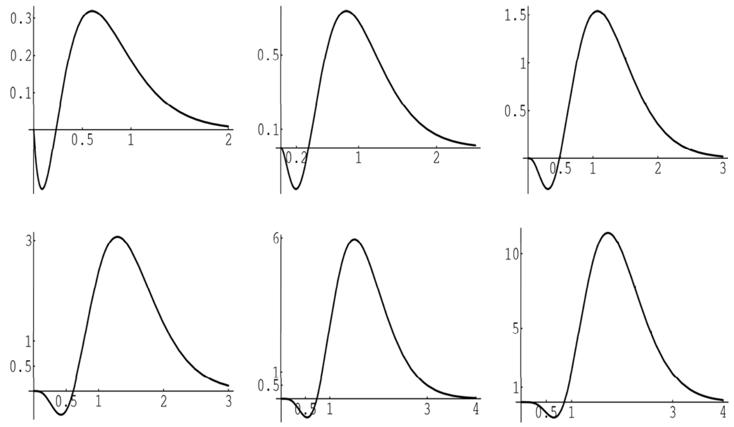

Figure 2: Influence functions for t=5

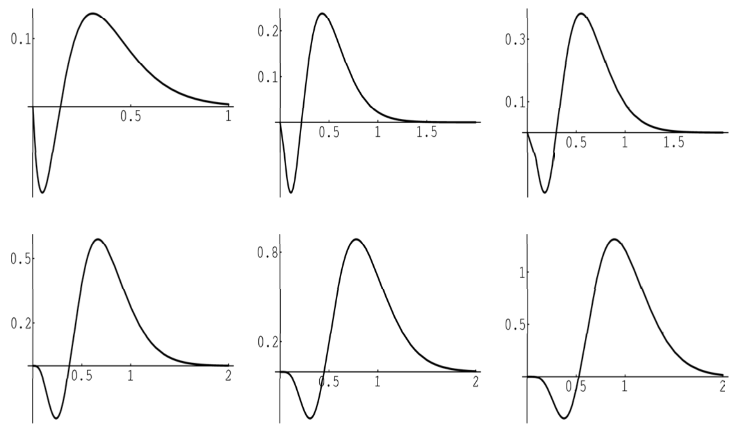

Figure 3: Influence functions for t=10

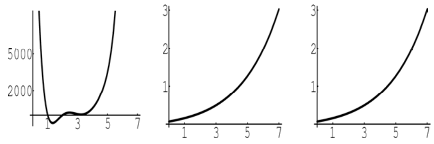

Figure 4: Norm of the derivative for $p = 1, q = \infty$ and $t = 1, 5, 10$

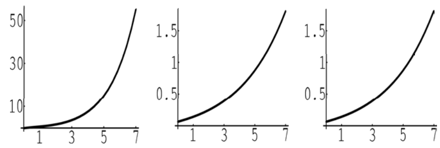

Figure 5: Norm of the derivative for $p = 2, q = 2$ and $t = 1, 5, 10$

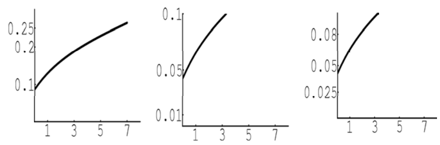

Figure 6: Norm of the derivative for $p = \infty, q = 1$ and $t = 1, 5, 10$

6 Conclusions and extensions

The aim of this paper is to illustrate notions and techniques of Bayesian local robustness in the context of problems that arise in Actuarial Science. Due to this procedure, new results were obtained.

Assuming that it is really difficult to quantify an expert's priori opinion in a single prior distribution, we suggest to measure the changes in the premium under infinitesimal changes in the prior. We have combined the tools of standard and local Bayesian robustness in order to show how the choice of the prior can have a crucial effect on the premium.

We believe that a prior distribution giving more weight to the region where the sample size is in conflict with the prior mean is necessary. This can be done, for example, by using a prior which can be built as a convex sum of two or more distributions.

The present study leaves some aspects open to question, which could be the subject of future study. First, the consideration of the exponential principle (Heilmann, 1989) allows the practitioner to choose the parameter which characterizes the risk aversion of the insurer. By using different values of this parameter, we probably may reduce the value of the norms. But new results are required to work with this premium principle. Second, since in Actuarial Science the mean and the mode are natural concepts, an actuary who has a good statistical training should not have any problem in assessing these characteristics on the risk parameter and its numerical values, based in his experience. The mode is impossible to incorporate to the problem treated in this paper, at least by now; conditions on moments can be possible using the methodology of Betrò *et al.* (1996) and Moreno *et al.* (2003). Third, linear perturbations on the prior have been treated in this paper, but non linear perturbations are possible, in the line of the work of Gustafson (1996).

References

Betrò, B., Meczarski, M. & Ruggeri, F. (1994). Robust Bayesian analysis under generalized moment conditions. *Journal of Statist. Plann. Inference.* **41**, 257–266.

Bühlmann, H. (1970). Mathematical Methods in Risk Theory. *Springer–Verlag, New York.*

Diaconis, P. & Freedman, D. (1986). On the consistency of Bayes estimates. *Annals of Statistics.* **14**, 1–67.

Gómez, E; Pérez, J., Hernández, A. & Vázquez, F. (2002). Measuring sensitivity in a Bonus–Malus System. *Insurance: Mathematics & Economics.* **31**, 1, 105–113.

Gustafson, P. (1996). Local sensitiviy of posterior expectations. *The Annals of Statistics.* **24**, 1, 174–195.

Gustafson, P. and Wasserman, L. (1995). Local sensitiviy diagnostics for Bayesian inference. *The Annals of Statistics.* **23**, 6, 2153–2167.

Martín, J; Ríos, D. & Ruggeri, F. (2003). Join sensitivity in Bayesian Decision Theory. *Test.* **12**, 1, 173–194.

Milne, R.D. (1980). Applied Functional Analysis. An Introductory Treatment. *Pitman Advanced Publishing Program.*

Moreno, E., Bertolino, F. & Racugno, W. (2003). Bayesian Inference Under Partial Prior Information. *Scandinavian Journal of Statistics.* **30**, 565–580.

Ruggeri, F. & Wasserman, L. (1993). Infinitesimal sensitivity of posterior distributions. *Canadian Journal of Statistics.* **21**, 195–203.

Address for correspondence: Emilio Gómez Déniz, Department of Quatitative Methods, University of Las Palmas de Gran Canaria, 35017–Las Palmas de Gran Canaria, Spain. E–mail: `egomez@dmc.ulpgc.es`